PHY293 Lecture #6

November 6, 2017
Clarifications: November 14, 2017

1. World-Lines view of Twin Paradox (Spacetime diagrams)
   - We can get another view of this by constructing what are called the world-lines of the various events
     - One axis has time in the stationary frame (Bob's clock)
     - The other axis has position or distance in Bob's (stationary) frame
     - Anna's journey is a line with slope 0.8 ($v = 0.8c$)
     - Carl's journey is a line with slope -0.8 ($v - -0.8c$ heading back towards earth)
     - Light signals are represented by lines with slope ± 1 on this plot ($v = c$)
   - Consider pictures/signals sent out towards planet every 10 years by Bob (not just first one).
     - The first of these will arrive just as Anna/Carl arrive at planet.
     - Bob will be 50 then (and will have already sent out 4 other pictures/signals)
     - Anna/Carl can also send signals every 10 years back to Bob. Slope -1 (or $-c$) on this plot
     - The first of these will reach Bob at age 30 (Work this out with Lorentz transformations as an exercise?)
     - From this signal Bob concludes that Anna is aging 1/3 as fast as he is (10 yrs vs. 30 yrs)
     - Anna is 30 when she reaches the planet and the first of Bob's signals reaches her. So she concludes that she is aging 3x faster than Bob (30 yrs vs. 10).
     - The problem is symmetric – at least for the out-bound leg: Both conclude that they are aging 3x faster than the other.
   - Now consider signals on the return journey
     - Carl receives an image of Bob as a 40-year old, when he is 40 (numerical coincidence!)
     - In general, Carl will receive signals 3x per decade on the return trip
     - Considers that Bob is aging 3x as fast as he/Carl is.
       - Carl ages 30 years on return journey. 9 of Bob's signals/images (90 years) arrive at Carl's ship during this time.
   - Consider the return signals (from Anna/Carl) to Bob.
     - Bob gets first signal when he is 30 (Anna is 10)
     - Bob gets second signal when he is 60 (Anna is 20)
     - Bob gets last signal from Anna (or first signal from Carl) at 90 (Anna/Carl are 30)
     - Then gets 3 signals between 90 and 100
     - Again concludes he is aging 3x faster than Anna during her out bound journey
     - But concludes he is aging 1/3 as fast during the last decade when Carl is signaling every 10 years during the return journey. Again the problem is symmetric.
   - Note that the sending of these signals is analogous to the peaks of a lightwave in the relativistic Doppler Effect
     - Rate at which signals received decreases by a factor $\frac{\sqrt{1+v/c}}{\sqrt{1-v/c}} = 3$ when source is **receding** from observer ($v = 0.8c$)
     - When source is approaching the observer the factor becomes: $\frac{\sqrt{1-v/c}}{\sqrt{1+v/c}} = 1/3$
     - Just to clarify, in lecture 4 (Oct 31 lecture notes) is that the above is the factor between the arrival time of the signals.
     - Signals arrive with a separation 3x longer when the source is receeding (either Bob or Anna, during outbound journey)
     - The signals arrive with a separation that is 1/3 as large during the inbound journey.
     - We also learned that the signal frequency difference is the inverse of the signal times so during the outbound journey the frequency changes by a factor $\frac{\sqrt{1-v/c}}{\sqrt{1+v/c}} = 1/3$ when source is **receding**
     - When source is approaching the observer the frequency of the signals increases a factor: $\frac{\sqrt{1+v/c}}{\sqrt{1-v/c}} = 3$
     - You can think about this either way, and should come to compatible conclusions

2. Four-vectors
   - Familiar with 3-vectors – positions in space who's lengths (magnitude squared) are invariant under spatial rotations.

- Examples include: position, velocity, momentum... (x, y, x) (v_x, v_y, v_z) etc.
- In space/time define four-vectors that include 'time' as 4th coordinate: (x, y, z, ct) NB. all components must have same units
- Any four component object (A_x, A_y, A_z, A_t) that transforms like (x, y, z, ct) – using Lorentz transformations is also four-vector.
- Classically spatial and time separation of two events are independent: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ and $\Delta t = t_2 - t_1$
- Not true in special relativity – time and spatial separations are mixed by the Lorentz transformations. Must be mixed if all observers agree on speed of light.
- Can define a Lorentz Invariant, that all inertial observers will agree on (4-dim analog of distance in 3-space)
- $A_t^2 - A_x^2 - A_y^2 - A_z^2 = A_t^2 - \vec{A} \cdot \vec{A}$ is the 4-d analog of the length of a vector in 3-space.
- Even dot product of two different 4 vectors is invariant: A and B defined as: $A_t B_t - A_x B_x - A_y B_y - A_z B_z = A_t B_t - \vec{A} \cdot \vec{B}$ (eg. $\vec{F} \cdot \vec{v} = P$ in three space).
- Check this by considering the dot product of the position vector with itself (the 4d analog of distance):

$$I = (ct)^2 - x^2 - y^2 - z^2$$

- Sub in Lorentz transformed quantities $x = \gamma(x' + \beta ct')$ and $t' = \gamma(t' + \beta/cx')$ and get I'

$$\begin{aligned}
I &= (\gamma ct' + \gamma\beta x')^2 - (\gamma x' + \gamma\beta ct')^2 - (y')^2 - (z')^2 \\
&= \gamma^2(ct')^2 + \gamma^2\beta^2(x')^2 \boxed{+\,2\gamma^2\beta(ct')(x')} \quad \text{cancel} \\
&\quad - \gamma^2(x')^2 - \gamma^2\beta^2(ct')^2 \boxed{-\,2\gamma^2\beta(ct')(x')} - (y')^2 - (z')^2 \\
&= \gamma^2(1-\beta^2)(ct')^2 - \gamma^2(1-\beta^2)(x')^2 - (y')^2 - (z')^2 \\
&= (ct')^2 - (x')^2 - (y')^2 - (z')^2 = I'
\end{aligned}$$

reminder

$$\boxed{\gamma^2(1-\beta^2) = \frac{1}{(1-\beta^2)}(1-\beta^2) = 1}$$

- The same in either frame, so true in **any** inertial frame (any v).

3. Specific example

- Consider two events:
  - Event 1: x' = 10m, y'=0m, z'=5m, t'= 3×10^{-8} s
  - Event 2: x' = 20m, y'=10m, z'=0m, t'= 6×10^{-8} s
- Already saw that $(ct)^2 - x^2 - y^2 - z^2$ was invariant so $(c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$ will be as well.
- The latter represents the (square of) the space-time interval between the two events.
- Classically we have the spatial separation and time difference and they are independently conserved. Special relativity requires that only the special combination (the space time interval) be conserved.
  - If the spacetime interval squared is negative: space-like
  - If the spacetime interval squared is positive: time-like
  - If the spacetime interval squared is zero: light-like ... why?
- Compute the invariant in S'. Only consider Δx and Δt parts because Δy and Δz parts identical in all frames

$$c^2(\Delta t')^2 - (\Delta x')^2 = (3 \times 10^8)^2(3 \times 10^{-8})^2 - (10)^2 = 9^2 - 10^2 = -19$$

- Can boost these to two different stationary frames
  - First has $v = 0.6c$, $\beta = 0.6$, $\gamma = 1.25$ (see slides)
  - Second has $v = 0.8c$ $\gamma = 1.6666$
- All three frames give I=I'(0.6c) = I'(0.8c) = -19.
- This interval is spacelike (negative) – two events are separated by enough distance there is no frame where they occur at the same point in space.
- If interval (squared) is positive, then some observers will see the two events at same point in space, just at different points in time (time-like)

- When the interval is 0, then only observers moving at speed of light can see the events overlap. Events can only be connected by a light beam.

4. Causally connected Spacetime events

- Events with space-like separations cannot be causally connected
- No information can get from one event to the other – would have to travel faster than c
- Lightcone delimits space time into regions that are:
  - Causally connected (inside the light cone, separated by time-like intervals)
  - Not causally connected (outside the light cone, separated by space-like intervals)
  - On the light cone – separated by intervals that can be covered only at the speed of light
- Two events, separated in spacetime by an interval that is larger than light can travel, cannot be causally connected.
  - Nothing that has occurred more than 100 light years away from us can have had any influence on our lives (yet) because the first information that they happened will only arrive 100 years after they happened. If the sun goes supernova, we won't find out about it until 8 minutes after it happens... But can never know it is coming...
  - Relativity also predicts/explains that events can be observed to happen in a different time order in different reference frames, but only if they are separated by a spacelike interval.
  - IF the interval is time-like $(c\Delta t)^2 > (\Delta x)^2$ then all observers must agree on the order of the events (not necessarily the time, or spatial separation, between them, but the cause-and-effect order).
- Causal effect Example
  - Bob/Bob Jr in stationary frame: Bob has a big idea and tells Bob Jr. about it.
  - Anna/Amy in moving frame: Amy next to Bob Jr. when Bob Jr. first hears of Bob's idea
  - Amy tells Anna about the idea ... as quickly as possible, given their separation.
    * As usual S and S' coincide at $t = t' = 0$
    * Bob transmits his idea to Bob Jr. at speed u_0 (show that $u_0 \leq c$ later)
    * Amy is 'right next' to Bob Jr. when he receives idea so he can pass it along instantaneously.
    * Amy then transmits the information to Anna also at u_0 – fastest communication channel for either of them.
  - Event 1: Bob's idea at $x = x' = 0$ and $t = t' = 0$
  - Event 2: Arrival of Bob's transmission to Bob Jr. (x_2) which is passed on to Amy instantly She is at x'_2
  - In S this occurs at $t_2 = x_2/u_0$
  - Use Lorentz transformations to find coordinates of second event in S'

$$x'_2 = \gamma(x_2 - vt_2) = \gamma(x_2 - v\frac{x_2}{u_0}) = \gamma x_2(1 - \frac{v}{u_0})$$

$$t'_2 = \gamma(t_2 - v/c^2 x_2) = \gamma(x_2/u_0 - v/c^2 x_2) = \gamma x_2(\frac{1}{u_0} - \frac{v}{c^2})$$

  - Amy immediately transmits information to Anna (again at u_0)
  - Event 3: Arrival of the information at the origin in S' (Anna)
  - This happens at $t'_3 = t'_2 + x'_2/u_0$
  - Using the Lorentz transformed coordinate predictions above we get:

$$t'_3 = \gamma x_2[(\frac{1}{u_0} - \frac{v}{c^2}) + \frac{1}{u_0}(1 - \frac{v}{u_0})] = \gamma x_2 \frac{2}{u_0}[1 - \frac{v}{c}(u_0/2c + c/2u_0)]$$

  - What we are looking for is if t'_3 is positive then causality is OK.
  - Anna and Bob were side-by-side at $t = t' = 0$ when Bob had his idea. So as long as Anna doesn't know about until sometime after $t' = 0$ then causality is OK
  - So what are the conditions for t'_3 being positive?
    * If $u_0 < c$ then t'_3 is always positive (see below)
    * If $u_0 > c$ then there is some $v < c$ where t'_3 is negative, which would mean that Anna receives word of Bob's big idea **before** the origin of her reference frame overlapped with Bob – ie. before Bob had the idea at all.
  - To get a negative t'_3 what is the condition on v?

- Square brackets must be less than 0 gives:

$$1-\frac{v}{c}(u_0/2c+c/2u_0)<0 \Rightarrow \frac{v}{c}(u_0/2c+c/2u_0)>1$$

- This in turn leads to $\frac{v}{c}>(u_0/2c+c/2u_0)$ which finally gives $\frac{v}{c}>\frac{u_0}{c}\frac{2}{1+u_0^2/c^2}$
- But if $u_0<c$ then second term is always bigger than 1 leading to the conclusion that t_3' can only be negative if $\frac{v}{c}>\frac{u_0}{c}$
- But we can't have $v>u_0$ because then Bob's idea wouldn't get to Bob Jr/Amy before Amy had already passed Bob Jr ... so she could never get the idea to transmit back to Anna.
- So for $u_0<c$, t_3' must be positive for any value of v.
- If, somehow $u_0>c$ then consider the construct:

$$(\frac{u_0}{c}-1)^2 \geq 0 \Rightarrow \frac{u_0^2}{c^2}-2\frac{u_0}{c}+1\geq 0 \rightarrow \frac{u_0^2}{c^2}+1 \geq 2\frac{u_0}{c} \Rightarrow 1 \geq \frac{u_0}{c}\frac{2}{1+u_0^2/c^2}$$

- Since the RHS of this last equality never exceeds 1 it means there is some v/c in the equation above for $v<c$ where t_3' is negative – and causality is violated.
- To summarise: Causality may be violated ($t_3'<0$) if $u_0>c$. So the fastest we can transmit Bob's idea is $u_0<c$.