Relativistic Energy and Momentum

Classically, the linear momentum of an object is given by the product of the object's mass and velocity: mu.

The requirement that the laws of physics be the same in inertial reference frames related by a Galilean velocity transformation implies that (classical) linear momentum is conserved (I won't prove this, or ask you to). Similarly:

The requirement that the laws of physics do not change with time (translations in time) requires that energy be conserved.

The requirement that the laws of physics not depend on rotations of the coordinate system results in the requirement that angular momentum must be conserved.

Noether's Theorem states that every symmetry in nature is associated with a conservation law, and vice versa.

Conservation of Momentum

For a closed system of particles, if the system is isolated there can be no net force and so

$$\vec{P}_{total} = \sum_i \vec{p}_i = \sum_i m_i \vec{u}_i = \text{constant}$$

It is straightforward to show that if $\vec{P}_{total}$ is conserved in one (Galilean) inertial reference frame it is conserved in any other (apply the Galilean velocity transformation -- for v along the x direction we can just look at the x component):

$$P'_{total} = \sum_i m_i u'_i = \sum_i m_i \left(u_i - v\right) = \sum_i m_i u_i - \sum_i m_i v$$

$$P'_{total} = P_{total} - v \sum_i m_i$$

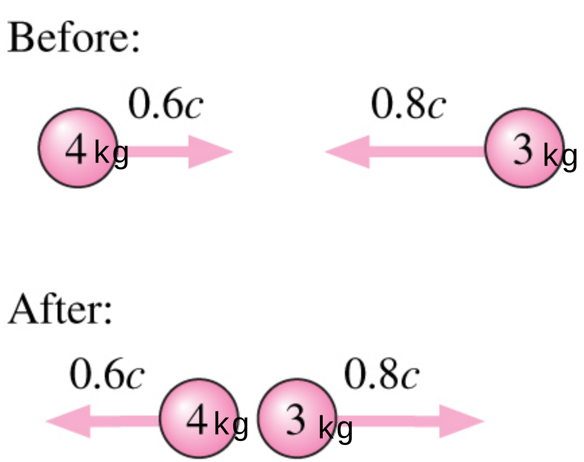
P'_{total} and P_{total} differ by a constant amount (v is constant and the masses do not change). So if one is conserved, so is the other.

Relativistic Momentum

HOWEVER: we know that the Galilean velocity transformation that we applied is NOT valid at velocities v close to c.

What happens if we apply a relativistic velocity transformation?

This is Example 2.8 in the text. For the collision illustrated below, apply the relativistic velocity transformation from S to S':



(a) Seen in frame S

Here the numbers in the circles are the masses of each object

Note that classically, this collision takes place the centre-of-mass frame (i.e. equal and opposite momenta).

For the moving frame S' we chose a frame moving to the right at 0.6c so that the left hand particle in the S frame (4) is at rest in S'.

$$u' = \frac{u - v}{1 - \left(uv / c^2\right)}$$

Classically $p = mu$ and so we have equal and opposite momenta in the initial state and so must also in the final state. Now use the (relativistic) velocity transformations to get the transformed momenta of the initial and final states and compare:

4 kg 3 kg

Before:

$$\frac{0.6c - 0.6c}{1 - \dfrac{(0.6c)\cdot(0.6c)}{c^2}} = 0 \qquad \frac{-0.8c - 0.6c}{1 - \dfrac{(-0.8c)\cdot(0.6c)}{c^2}} = -0.946c$$

After:

$$\frac{-0.6c - 0.6c}{1 - \dfrac{(-0.6c)\cdot(0.6c)}{c^2}} = -0.882c \qquad \frac{0.8c - 0.6c}{1 - \dfrac{(0.8c)\cdot(0.6c)}{c^2}} = 0.385c$$

Before: $\displaystyle\sum_i m_i u_i' = 4(0) + 3(-0.946c) = -2.84c$

After: $\displaystyle\sum_i m_i u_i' = 4(-0.882c) + 3(0.385c) = -2.38c$

Clearly something is wrong. If we trust our (relativistic) transformation then we are left only with the definition of linear momentum.

(i.e. as the source of the problem)

Relativistic Momentum Cont'd

Relativistically, the correct expression for the momentum involves (unsurprisingly, I think) the Lorentz boost factor:

$$\vec{p} = \gamma m \vec{u}$$

This reduces to the classical expression in the non-relativistic limit in which $\gamma \rightarrow 1$ (as required).

In light of this, note that in the example on the previous slide, the collision is not (relativistically) taking place in the centre-of-mass frame (more properly called the centre-of-momentum frame) since the Lorentz boost factors are different for the two objects (so their relativistic momenta are not equal and opposite).

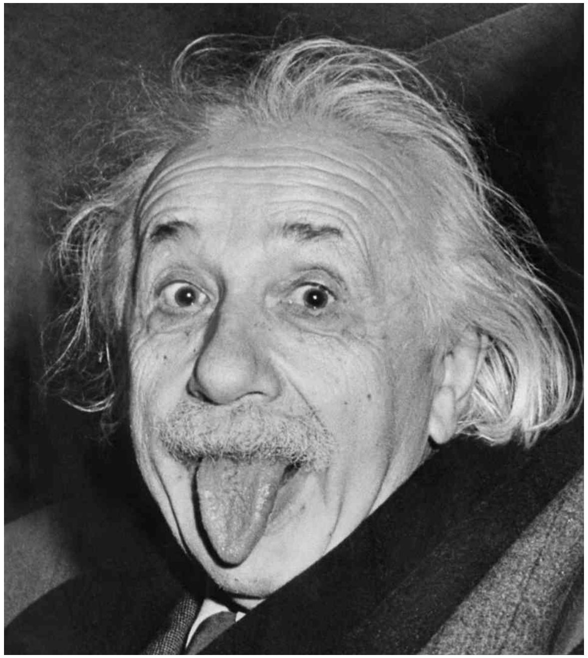
Relativistic Energy

The most famous equation in physics is:

$$E = mc^2$$

Most physicists would write this as*:

$$E = \gamma m_0 c^2$$



Here m_0 is the mass of the object in a frame in which it is at rest. [We will discuss massless particles later on]

This is arguably the only frame in which it makes sense to define this (as for a particle's lifetime) since everyone can agree on this. I will just write this as m from now on [to always mean rest mass].

A particle at rest (so a non-relativistic particle) thus has a (rest) energy of $E=mc^2$. This has no classical analogue.

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* The more famous formulation relies on the idea of a relativistic mass, e.g. γm, which most physicists dislike (see pg 38)

Relativistic Energy Cont'd

The rest energy is the "internal" energy (of the particle or object), whatever form this might take. This, however, can be converted into kinetic energy (as we shall see).

The kinetic energy of a particle is its total energy minus the rest energy

$$K.E. = \gamma mc^2 - mc^2 = \left(\gamma - 1\right)mc^2$$

Consider the energy of a non-relativistic object (so u << c)

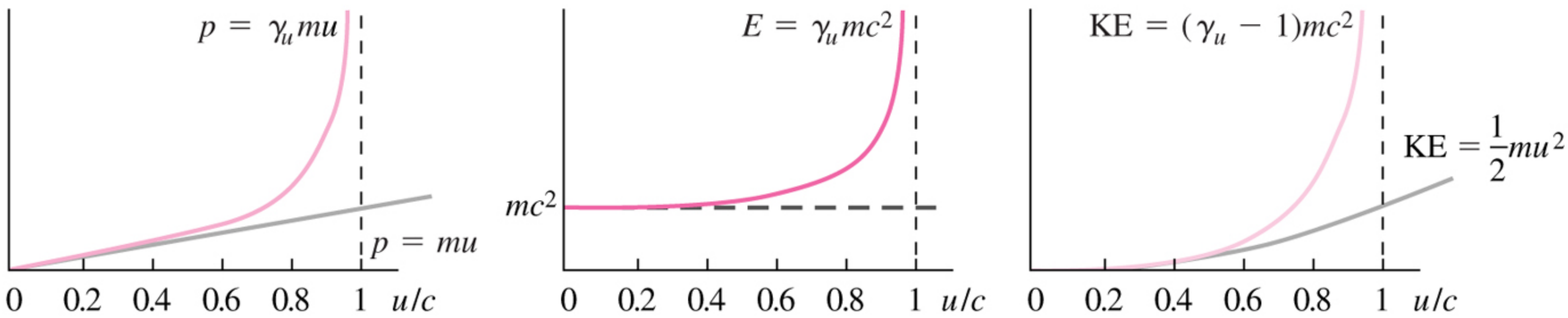
$$\gamma mc^2 = mc^2\left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx mc^2\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{u^2}{c^2}\right)\right] = mc^2 + \frac{1}{2}mu^2$$

[This approximation is just via the binomial expansion]

$$\left(1+x\right)^m = 1 + \sum_{n=1}^{\infty} \frac{m(m-1)(m-2).....(m-n+1)}{n!}x^n \quad \left|x\right| < 1$$

and the classical expression for the kinetic energy is obtained.

Kinematic quantities as a function of velocity ($\beta = u/c$)



Massless particles travel at the speed of light, c.

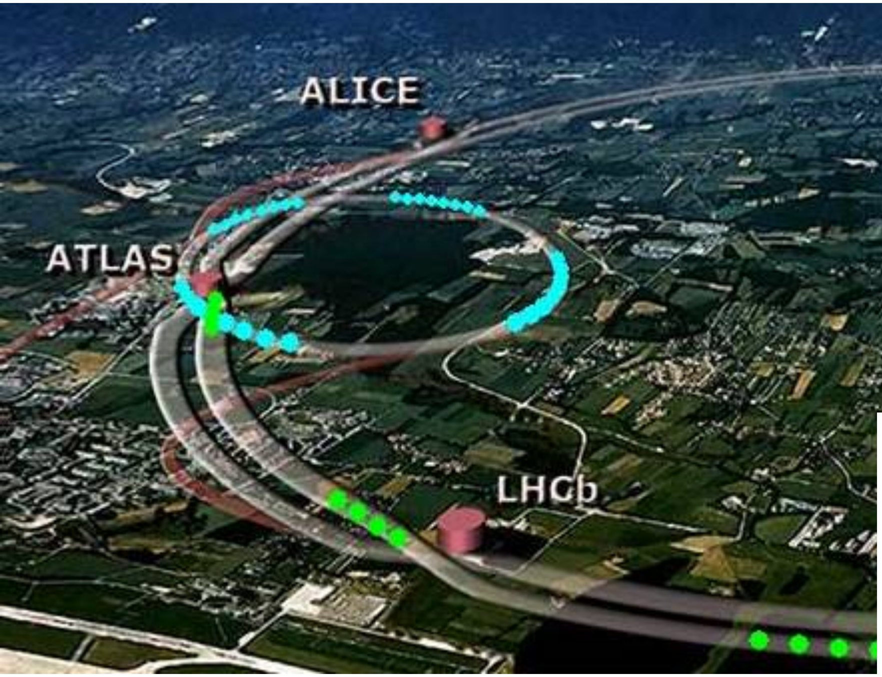
Massive particles are limited to speeds $u < c$.

Energy and momentum both grow steeply as a function of u as this approaches c.

Note that $E / mc^2 \; (= \gamma mc^2 / mc^2 = \gamma)$ provides a measure of how relativistic a particle is. We used this when discussing cosmic-ray muons in Lecture 3.

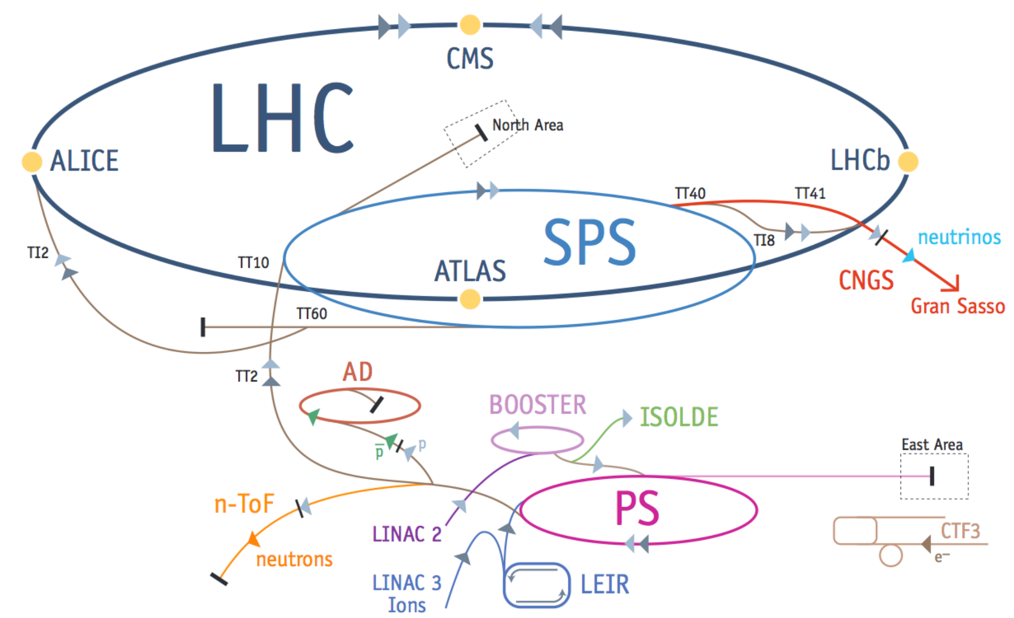
Particle Colliders

Convert the (some or all of) the kinetic energy of high-energy beam particles into rest energy of particles in the final state.



2012: 4000 GeV protons

27km circumference



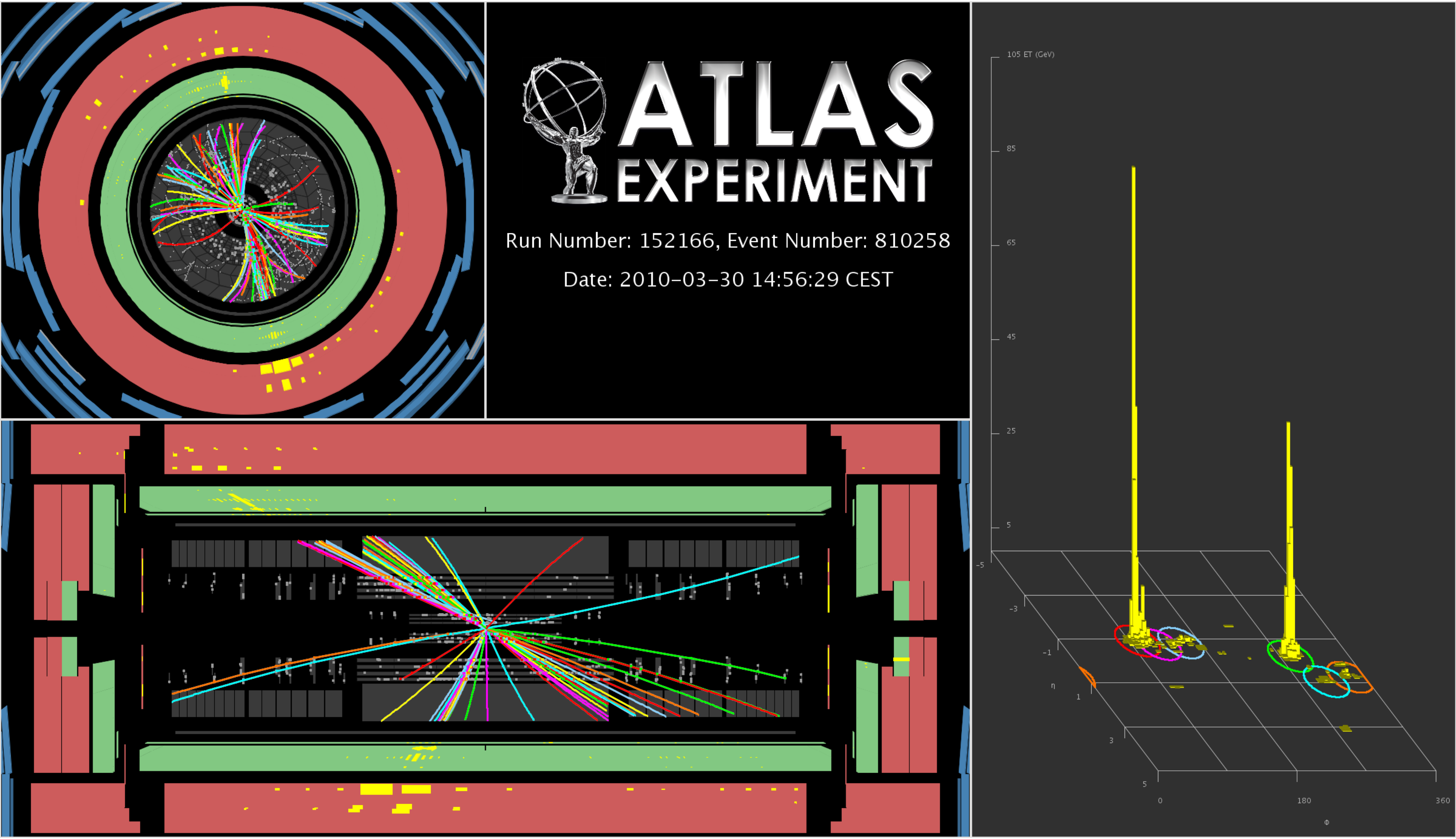
LHC Beam Energy

$$\gamma = E / m_p c^2 = 4000[\mathrm{GeV}] / 0.989[\mathrm{GeV}] \approx 4000$$

$$\gamma = (1 - \beta^2)^{-1/2} \approx 4000 \quad \Rightarrow \quad \beta = .99999997$$

$$\frac{27 \times 10^3 m}{3 \times 10^8 m / s} = .00009s \quad \Rightarrow \quad 11{,}000 \text{ orbits / s}$$

Particle Colliders (The LHC)



Nuclear Power

Nuclear reactors work on a similar principle. But these convert rest-energy into kinetic energy (which e.g. heats water that turns turbines).

In fact, the radioactive processes used to generate nuclear power convert only a small amount of rest energy into kinetic energy.

If we could really access all the energy in some object, energy would not be a problem. For instance, a penny weights about 2g. How much energy does this correspond to?

$$E = mc^2 = \left(2\times 10^{-3}kg\right)\left(3\times 10^{8}ms^{-1}\right)^2 = 18\times 10^{13}kg\cdot m^2/s^2 = 1.8\times 10^{14}J$$

A Watt is a Joule/s, so a kilowatt-hour (kwh) is $\left(10^3 J/s\right)\cdot\left(3600s\right) = 3.6\times 10^6 J$

So the energy stored in the penny is equivalent to

$$\frac{1.8\times 10^{14}J}{3.6\times 10^6 J/kwh} = 50\times 10^6 kwh$$

A kg of ^{235}U releases 7.2 x 10^{13} *J*

The energy-momentum four-vector

The four-vector that is most useful for solving problems in relativistic dynamics is the energy momentum four-vector.

$$\left(p_x, p_y, p_{z,}, \frac{E}{c} \right)$$

This must (by definition) transform in the same way as the position-time four-vector, so :

$$p_x' = \gamma \left(p_x - \beta \frac{E}{c} \right), \;\; \frac{E'}{c} = \gamma \left(\frac{E}{c} - \beta p_x \right), \;\; p_y' = p_y, \;\; p_z' = p_z$$

What is the associated Lorentz invariant ?

$$A_t^2 - A_x^2 - A_y^2 - A_z^2 = \left(\frac{E}{c} \right)^2 - p_x^2 - p_y^2 - p_z^2 = \left(\frac{E}{c} \right)^2 - p^2$$

The energy-momentum four-vector

$$\left(p_x, p_y, p_{z,}, \frac{E}{c}\right): \quad A_t^2 - A_x^2 - A_y^2 - A_z^2 = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2 = \left(\frac{E}{c}\right)^2 - p^2$$

We had that $E = \gamma mc^2, p = \gamma mu$ so this becomes:

$$\left(\frac{E}{c}\right)^2 - p^2 = \gamma^2 m^2 c^2 - \gamma^2 m^2 u^2 \ = \gamma^2 m^2 c^2 \left(1 - (u/c)^2\right) = \gamma^2 m^2 c^2 \left(1 - \beta^2\right) \ = m^2 c^2$$

Here m is the mass of the particle described by the four-vector, or the *invariant mass* of the system of particles (if that is what is described).

The invariant mass is a very useful quantity because it is the same before or after any interaction (particle decay, collisions).

A note on massless particles

We will discuss photons in more detail when we begin discussing quantum mechanics next week. But for the time being:

On the previous slide we had $\left(\frac{E}{c}\right)^2 - p^2 = m^2c^2$.

Rearranging this we obtain $E^2 = p^2c^2 + m^2c^4$ which is referred to as the relativistic energy-momentum relationship.

For m=0 this becomes $E^2 = p^2c^2$ or $E = \left|\vec{p}\right|c$.

But what about $E = \gamma mc^2 = \dfrac{mc^2}{\sqrt{1 - u^2 / c^2}}$. Note that for m=0, if the

velocity u = c this is 0/0 (so undefined). This is just an observation. Don't make too much of it. But there are no inconsistencies here, as long as massless particles travel at the speed of light.

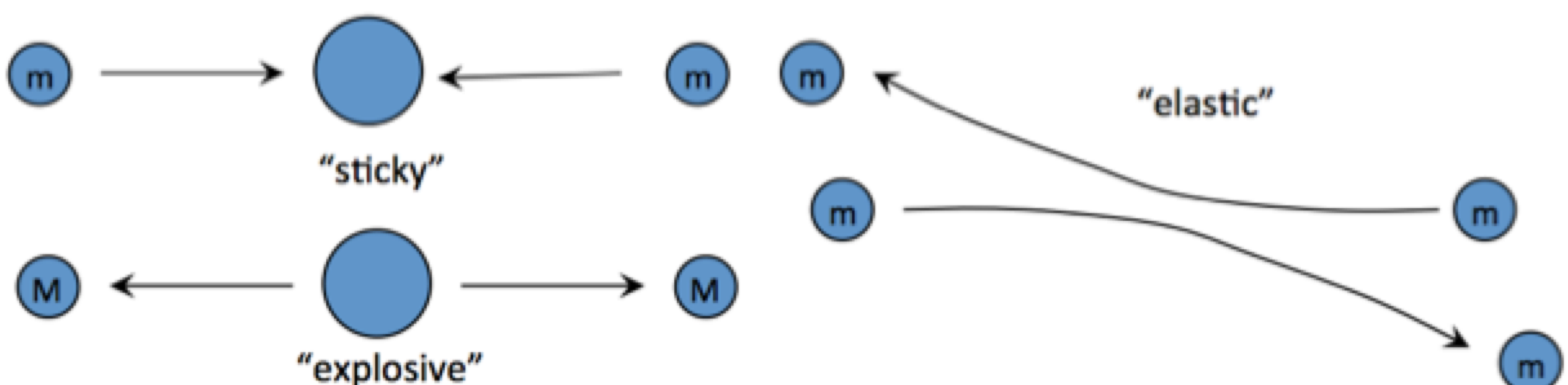
Interactions (in a generic form)

Different classes of processes (interactions): look classically, then relativistically: Consider the process $A+B \rightarrow C+D$

Classically what quantities are conserved in this collision process ?

- Mass: $M_A + M_B = M_C + M_D$
- Momentum: $\vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D$ (i.e. three-momentum)
- Kinetic energy may be conserved (elastic collision).

Consider three "types" of processes:



Conserved Quantities in Collisions

The quantities conserved in relativistic processes, are somewhat different than those conserved in the classical case since rest energy can be converted into kinetic energy, and vice versa.

Classically:

a) Energy and momentum are always conserved

b) Kinetic energy may be conserved (elastic collision)

c) Mass is always conserved

Relativistically:

a) Energy and momentum are always conserved

b) Kinetic energy MAY or MAY NOT be conserved

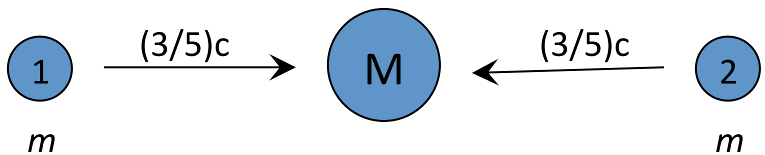
c) Mass MAY or MAY NOT be conserved

In a given process, these are either both conserved or both violated

Collisions where kinetic energy is conserved are referred to as "elastic". Those where it is not are referred to as "inelastic".

Example: Relativistic collision

Two particle of mass m and equal and opposite velocities of 0.6c collide to form an object at rest (net momentum is zero before and after the collision).



What is the mass M of the final state?

[Note: 1,2 here are just object indices]

Conservation of energy: $E_1 + E_2 = E_M = 2E_m$

Conservation of Momentum: $\vec{p}_1 = -\vec{p}_2$ [So final state momentum is 0]

How do I solve for M?

The final energy is $E = Mc^2$

The initial energy is $2E_m = 2\gamma mc^2$

$$M = 2\gamma m$$

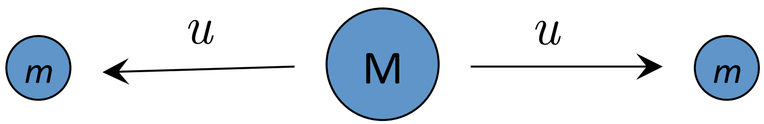
$$M = \frac{2m}{\sqrt{1-(3/5)^2}} = \frac{2m}{\sqrt{16/25}} = \frac{5}{2}m$$

which is > 2*m* (as it must be)

The kinetic energy in the initial state has been turned into mass energy in the final state

Relativistic Processes (Particle Decay)

In this example we instead have a particle of mass M decaying to two lighter particles of mass m. These will be emitted with equal and opposite momenta (and hence velocity).



What is the final state velocity u?

Conservation of energy: $Mc^2 = 2\gamma mc^2 \;\; \Rightarrow M = 2\gamma m = \dfrac{2m}{\sqrt{1-u^2/c^2}}$

$$\frac{M^2}{4m^2} = \frac{1}{1-u^2/c^2} \qquad 1-u^2/c^2 = \frac{4m^2}{M^2} \qquad u^2/c^2 = 1-\frac{4m^2}{M^2}$$

thus, $\displaystyle u = c\sqrt{1-\frac{4m^2}{M^2}} = c\sqrt{1-\left(\frac{2m}{M}\right)^2}$

Note that this makes sense only for *M > 2m*

In this case, the energy in the initial state is all in the form of mass energy, while the final state has both mass energy and kinetic energy.

Invariant Mass (Reminder)

For a particle or system of particles the energy momentum four-vector is:

$$P = \left(p_x, p_y, p_{z,}, \frac{E}{c} \right):$$

The "square" of this, $P^2 = P \cdot P$ (as defined previously) is a Lorentz invariant

$$P^2 = P \cdot P = P_t^2 - P_x^2 - P_y^2 - P_z^2 = \left(\frac{E}{c} \right)^2 - p_x^2 - p_y^2 - p_z^2 = \left(\frac{E}{c} \right)^2 - p^2 = M^2 c^2$$

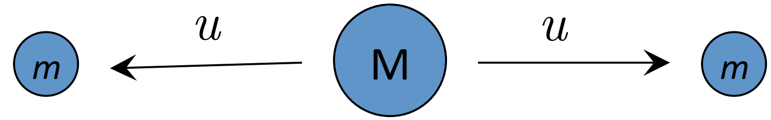
For a single particle, the M corresponds to the particle's mass. For a system of particles this represents the *invariant mass* of the system.

If a particle of mass M decays into a final state consisting of *N* particles, the invariant mass of the N-particle system equals the mass of the decaying particle.

Note that this is not really discussed in the textbook

The Use of Lorentz Invariants

Recall the following problem:



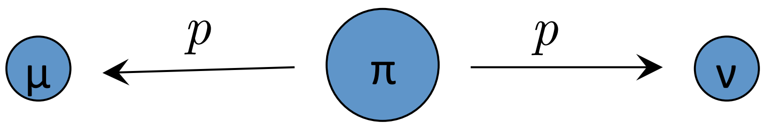
In four-vector notation, we can write the ***total*** four-momentum P in the initial and final states and make use of the fact that the invariant P^2 is the same before and after. By P^2 here, I mean the Lorentz-invariant we defined using the energy-momentum four-vector:

$$\left.\begin{array}{lll} \text{Initial state:} & (0,0,0,Mc) & P^2 = M^2c^2 \\ \text{Final state:} & (0,0,0,\, 2\gamma mc) & p^2 = 4\gamma^2 m^2 c^2 \end{array}\right\} \quad M^2c^2 = 4\gamma^2 m^2 c^2 \;\; \text{yields } M=2\gamma m \text{ as before}$$

e.g. $\left[\, (\vec{p},\gamma mc) + (-\vec{p},\gamma mc) \,\right]$

This is not a great simplification relative to the calculation that we did before. However, consider the case where the two final state particles have different masses. For example, let's consider the decay of a pion (π) into a muon (μ) and a neutrino.

Pion Decay Kinematics



For the masses we have $m_\pi > m_\mu$ and $m_\nu=0$. Find the momentum p.

In the initial state we have: $P_{tot} = P_\pi = \left(0,0,0,m_\pi c\right)$

In the final state we have $P_{tot} = P_\mu + P_\nu \quad \left[P_\mu = \left(-\vec{p}, \frac{E_\mu}{c}\right), P_\nu = \left(\vec{p}, \frac{E_\nu}{c}\right) \right]$

Conservation of energy and momentum means that $P_\pi = P_\mu + P_\nu$

or $P_\mu = P_\pi - P_\nu$. Squaring both sides (and using $P_X^2 = m_X^2 c^2$) we get

$$P_\mu^2 = P_\pi^2 + P_\nu^2 - 2P_\pi \cdot P_\nu$$

$$m_\mu^2 c^2 \qquad m_\pi^2 c^2 \qquad m_\nu^2 c^2 = 0$$

$$\vec{p}_\pi = 0$$

$$-2\left[\left(m_\pi c\right)\left(p_\nu\right) - \cancel{\vec{p}_\pi} \cdot \vec{p}_\nu\right] = -2m_\pi p_\nu c = -2m_\pi p_\mu c$$

$$\frac{E_\nu}{c} = \frac{p_\nu c}{c} = p_\nu$$

$$p_\mu = p_\nu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} c$$

see the discussion of massless particles from last time

Try doing this problem ***without*** the use of Lorentz invariants