PHY293 Lecture #14

November 24, 2017

1. More on matter waves
   - Last time began to explore DeBroglie's hypothesis that $\lambda = h/p$ for all particles
   - Saw how this was confirmed by Davisson/Germer experiment
   - Didn't discuss the velocity or frequency of these waves
     - As with EM waves these also have $\nu = E/h$
     - Can re-write wavelength as wave-number $k = 2\pi/\lambda$
     - Similarly can re-write $E = h\nu$ with $\nu = \omega/2\pi$ so $E = \hbar\omega$ where $\hbar \equiv h/2\pi$
     - Also $p = h/\lambda = \hbar k$
     - $\hbar = h/2\pi = 1.05 \times 10^{-24}$ J $\cdot$ s $= 6.59 \times 10^{-16}$ eV s
   - With these definitions $v = \nu\lambda = \frac{E}{h} \cdot \frac{h}{p} = \frac{E}{p} = \frac{\hbar\omega}{\hbar k} = \frac{\omega}{k}$
   - Saw this with Prof. Grisouard (King Ch 5) – phase velocity
   - This is is the speed of individual waves, but not necessarily the velocity a particle.
   - To represent a localised particle we have to construct a wave-packet (cf. waves part of course last few lectures)
   - This wavefunction represents a particle that has a higher probability to be somewhere near the centre of the packet in space ($|\Psi(x)|^2$ gives probability to find the particle at some x)
   - Making a wavepacket like this requires the super-position of many waves of different wavelengths
   - What is the velocity of such a packet?
   - The wiggles in the packet travel at the phase velocity of the individual waves
   - The overall envelope travels with the group velocity $\Rightarrow$ we interpret the group velocity as the speed the particle is moving
   - See, for example Prof. Grisouard's simulation from earlier this term.

2. Spatial Localisation of Matter Waves
   - For a single wavelength/frequency wave: $A\sin(kx - \omega t)$
     - For such a wave the momentum is **exactly known** ($p = \hbar k$)
     - But the position is **completely unknown** – the wave has constant amplitude (so amplitude^2 = constant) out to $x = \pm\infty$
     - Technically not constant for $\sin$ wave, but if formulated at e^{ikx} then $||^2$ is really constant everywhere.
   - Consider what happens if we try to improve the localisation
     - Single frequency $\Rightarrow$ position completely unconstrained/unpredicted
     - Localise better $\Rightarrow$ position better known, but superpose several wavelengths $\Rightarrow$ momentum gets some uncertainty
     - To really pin down the position $\Rightarrow$ have to include many more wavelengths/momenta in the packet
     - See section 4.7 for details on making packet. Or play with Prof. Grisouard's python scripts that generates wave packets
   - Conclusion: The more we know about position reported by $\Psi(x)$, the less we know about the momentum because the packet contains important contributions from a wide(r) range of wavelengths (and $p = h/\lambda$)

3. Energy and Time measurements
   - In Quantum Mechanics there is a similar 'tension' between measuring energy and the time you take to do it
   - To be sure you've observed all the wave frequencies that are contributing to the QM representation of the energy ($E = \hbar\omega$) you have to measure for very long times (otherwise you might miss contributions from very low frequencies)
   - If you measure for a finite period of time (Δt) then you are left with some uncertainty in your energy measurement (ΔE)
   - The **Heisenberg Uncertainty Principle** states:

$$\Delta x \Delta p \geq \hbar/2 \qquad \Delta E \Delta t \geq \hbar/2$$

   - For momentum these uncertainty relationships applies to each dimension independently:

$$\Delta x \Delta p_x \geq \hbar/2 \qquad \Delta y \Delta p_y \geq \hbar/2 \qquad \Delta z \Delta p_z \geq \hbar/2$$

4. The Heisenberg Uncertainty Principle

   - The position/momentum version simply states that we can never simultaneously know the position and its corresponding momentum (along that axis) with infinite precision
   - As with generating quantum interference patterns, this is a statement about **what can be measured**
     - When we collapse wavefunction we pick out one of the wavelengths – measure one with probability $|\Psi(\lambda)|^2$ (measure only one of momenta present)
     - Measure position of the particle with some probability given by $|\Psi(x)|^2$ (measure only one of the possible positions)
   - If we are going to try to setup a system that has high probabilities over a narrow range of x, then we are going to see any one of a wide range of wavelengths (momenta).
   - Quantum mechanics only makes probabilistic predictions of things that can be observed
   - Those are subject to the uncertainty principle

5. Single slit diffraction: Example of Uncertainty principle

   - Initially have a plane wave incident from the left: perfect precision on p_y (y is 0, wave is travelling in x direction only) but know nothing about y position – equal probability to find electrons at any y
   - As the electrons (their wavefunction) passes through the slit we gain information about where the electrons can be (only in a region of y given by Δy – the width of the slit
   - But the lose precision on the y momentum, the width of the diffraction pattern gives some measure of the range of p_y that the electrons gain by passing through the slit
   - As we saw from single slit interference last week $\Delta y \sin\theta = \lambda$ (i.e. distance to first minima in single slit pattern is Δy
   - If we take Δp_y to be the typical spread observed in p_y: $\Delta p_y = p_y = p\sin\theta = \frac{p\lambda}{\Delta y}$
   - Subbing in $p = h/\lambda$ give s $\Delta p_y = \frac{h}{\lambda}\frac{\lambda}{\Delta y}$ which can be rearranged to give $\Delta p_y \Delta y \approx h$
   - Note that this is greater than $\hbar/2$ (by about a factor of 12).
   - The uncertainty principle is the theoretical minimum. In this case our estimate of Δp_y, based on the first diffraction minimum is something of an overestimate.
   - Strictly speaking we'd want the 'one sigma' value for a Gaussian fit (which is probably a little smaller). Even the fact that Δy is taken to be the slit width is an overestimate.
   - What we were trying to get at here was the order of magnitude (governed by h or $\hbar$).

6. Uncertainty Principle and Measurement

   - If we had an electron somewhere along the x axis how would we determine it's position?
   - Scatter a photon off of it, and detect the scattered photon (and/or scattered electron)
     - But in doing that the electron would recoil, changing its position and momentum
     - eg. in Compton scattering that we discussed last week
     - Uncertainty principle arises because measuring something about system (collapsing wavefunction) disturbs system.
     - QM says there is fundamental limit to how small these disturbances can be
     - Provides us with a way of calculating them.

7. Examples of Uncertainty Principle in Action (skipped this in both classes – please take note)

   - Show that there is a relationship between the two version of the uncertainty principle
     - For a non-relativistic electron moving along the x axis with momentum p_x it has kinetic energy

$$E = mv_x^2/2 = p_x^2/2m \Rightarrow dE = 2p_x dp_x/2m = \frac{p_x}{m}dp_x$$

     - So if there is some uncertainty on p_x, Δp_x, the the uncertainty on E is going to be

$$\Delta E = \frac{p_x}{m}\Delta p_x \Rightarrow \Delta E = v_x \Delta p_x$$

     - Imagine trying to measure where this electron by shining a photon beam off of it.

- Suppose it takes Δt to measure this. Electron moving at v_x so a measurement that lasts Δt will incur a position uncertainty $\Delta x = v_x \Delta t$
- We can plug this back into the expression for ΔE: $\Delta E = \frac{\Delta x}{\Delta t}\Delta p_x$ which we can rearrange to see that $\Delta E \Delta t = \Delta x \Delta p_x$
- Having seen that there is a fundamental limit to $\Delta x \Delta p_x \geq \hbar/2$ we now see that $\Delta E \Delta t \geq \hbar/2$ also

- Hydrogen Atom
  - A hydrogen atom has $r = 0.1$ nm, implies that the electron orbit extends this far from the nucleus
  - Also say we don't really know where the electron is, but it must be somewhere around the nucleus with $\Delta r \approx 0.1$ nm
  - What does the uncertainty principle tell us about how precisely we can know the momentum (or speed) of the electron

$$\Delta r \Delta p_r \geq \hbar/2 \Rightarrow \Delta p_r \geq \frac{6.626 \times 10^{-34}\ \mathrm{J \cdot s}}{2\pi \cdot 2 \times 10^{-10}\ \mathrm{m}} = 5.3 \times 10^{-25}\ \mathrm{kgm/s}$$

$$p_r = m v_r \Rightarrow \Delta v_r = \Delta p_r / m \geq 5.8 \times 10^5\ \mathrm{m/s}$$

  - Typical speed we might measure for an electron bound to hydrogen atom can't be much smaller than this
    - Might be a factor of 1.5 or 3, but **not** 10 or 100x smaller
  - Electron is actually bound in hydrogen atom with 13.6 eV (amount of energy it takes to separate electron and proton).
  - What velocity does this correspond to

$$13.6\ \mathrm{eV} = mv^2/2 = 1/2(511 \times 10^3\ \mathrm{eV}/c^2)v^2 \Rightarrow \frac{v^2}{c^2} = \frac{27.2\ \mathrm{eV}}{511000\ \mathrm{eV}} \Rightarrow \frac{v}{c} = 0.0073$$

  - So this gives $v = 2.2 \times 10^6$ m/s ($< 0.01c$ so non-relativistic approx ok).
  - The uncertainty principle says that the electron velocity should be bigger than about 5.8×10^8 m/s
  - The two are consistent
  - This is an example where the uncertainty principle gives us an order of magnitude estimate of what to expect ... even if we don't know how to predict the 13.6 eV exactly, we know it can't be much less than a few eV.

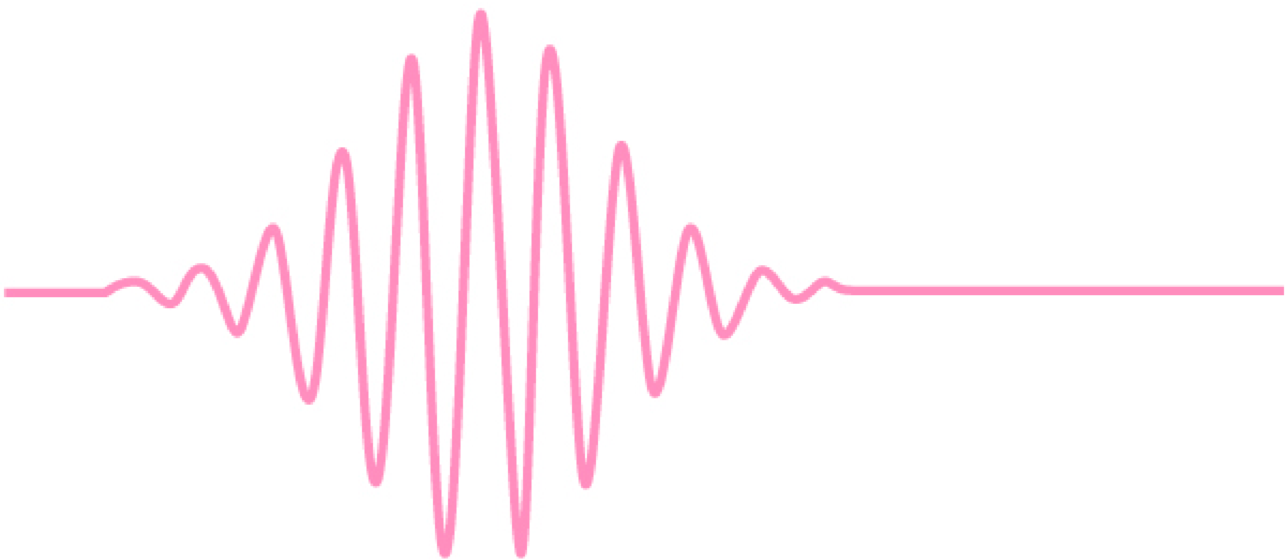
- What about λ for $p = 0$? (See Example 4.5 in text, maybe come back to this in next lecture?)
  - Actually the uncertainty principle prevents you from having an object with $p \equiv 0$
  - You would have no information about where the particle was
  - Consider locating a macroscopic object (one that you'd think you could get to $p = 0$
    - Using visible light ($\lambda \approx 550$ nm) then you can (maybe?) localise a grain of sand (1 mg) to 1 μm (about the smallest distance it could travel and you might still think it is stationary

$$\Delta x \Delta p_x \geq \hbar/2 \Rightarrow \Delta v_x \approx \frac{\hbar/2}{m \Delta x} \approx 10^{-22}\ \mathrm{m/s}$$

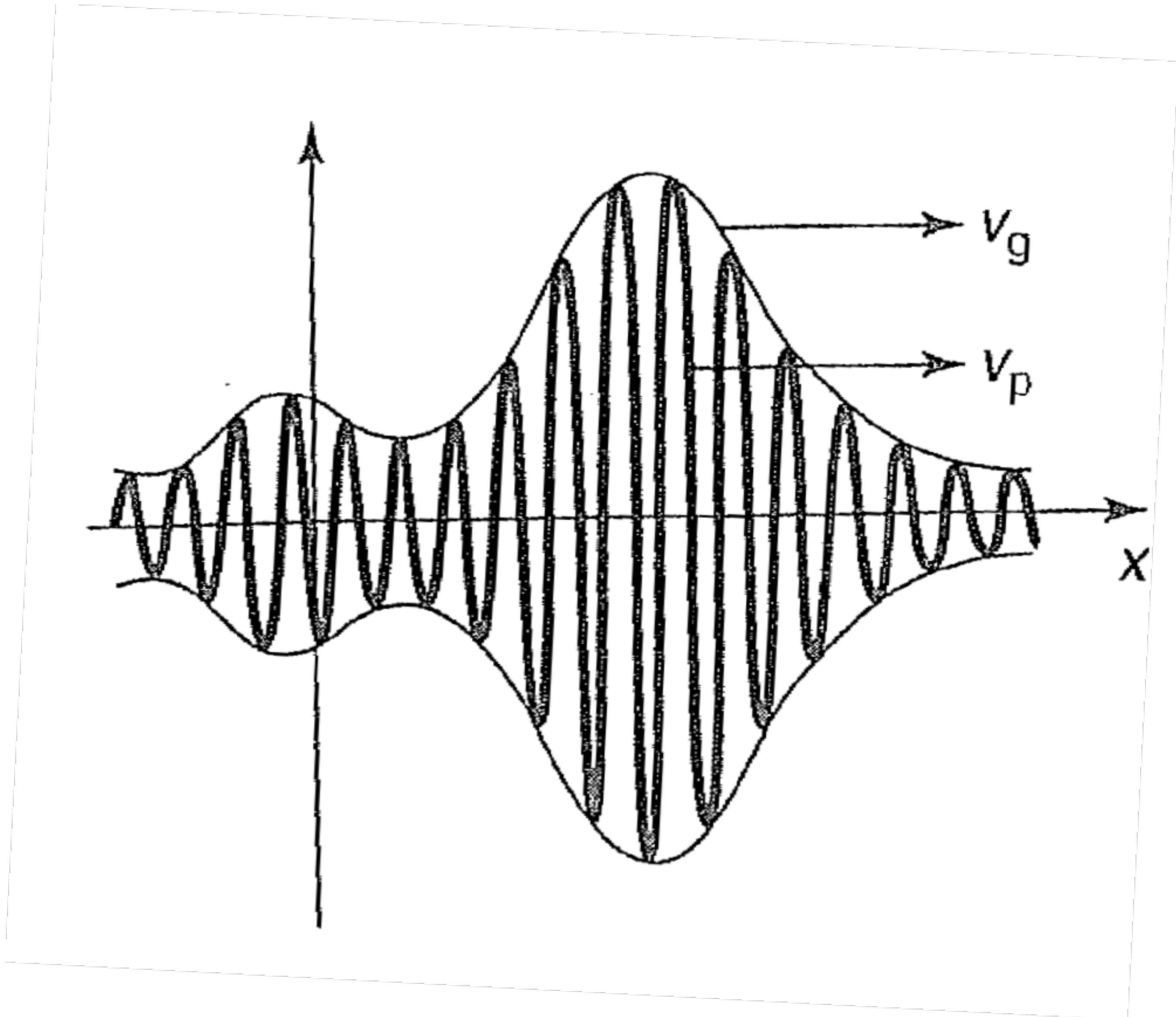
    - How long would it take to travel 1 μm at this speed? 10^{16} s (3 million centuries)
    - For all practical purposes the grain of sand would be stationary, but it would still obey $\Delta x \Delta p$ uncertainty principle
  - Even the tiniest classical object (could go down another factor of 1000 to bacteria and proteins and still come to similar conclusions) can appear to be completely stationary and still have a wavelength short enough to ensure particle-like behaviour.

- Particle Masses and Lifetimes (probably won't come back to this in next lecture)
  - Unstable, strongly interacting particles have lifetimes O(10^{-23}s).
  - If we determine the rest energy of a particle, can only measure energy/mass as long as the particle hasn't decayed
  - $\Delta E \Delta t \geq \hbar/2 \Rightarrow \Delta E \geq \hbar/2 \frac{1}{10^{-23}} = 10^{23} \frac{4.14 \times 10^{-15}\ \mathrm{eV \cdot s}}{4\pi} = 33$ MeV
  - This ΔE represents fundamental uncertainty on how precisely can determine the rest energy (a.k.a. mass) of a particle
  - Given a large sample of particles, even if you measure each very precisely, get a distribution who's mean we call 'the mass of particle', but also a finite spread, that we call the 'width of the particle': comes from the fact that we are limited in the amount of time we have to measure the particle's mass, before it decays.

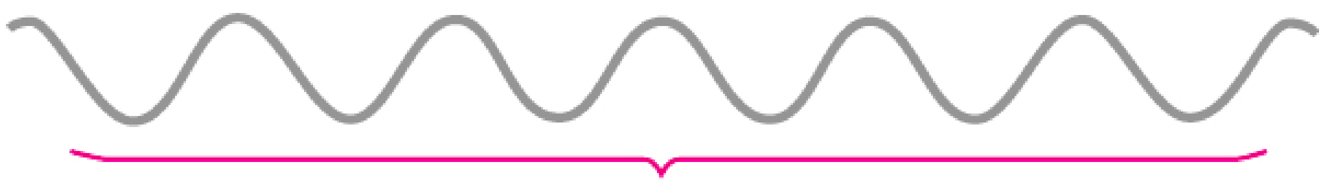
Wavepacket



Griffiths (QM text Fig 2.11)



Plane Wave

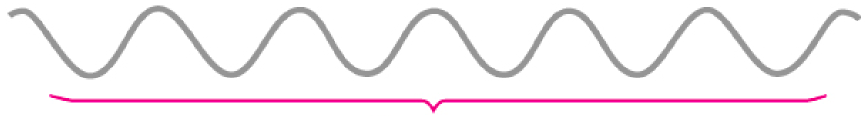


$\Delta x = \infty$

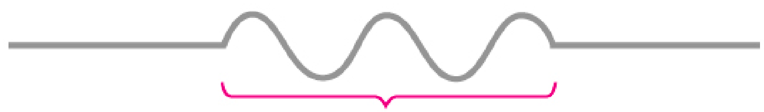
position completely unknown;

λ and p well defined

Localisation of Wavefunction



$\Delta x = \infty$
position completely unknown;
λ and p well defined



Δx finite
position better known;
λ and p less well defined

Δx small
position even better known;
λ and p even less well defined

Single Slit Unceratinty