PHY293 Problem Set 6 Solutions

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Harris 3.11

(a)

For small frequencies: $e^{\frac{hf}{k_B T}} \approx 1 + \frac{hf}{k_B T}$. Using this approximation in Planck's spectral energy density:

$$\begin{aligned}
\frac{dU}{df} &= \frac{hf}{e^{\frac{hf}{k_B T}} - 1} \frac{8\pi V}{c^3} f^2 \\
&\approx \frac{hf}{1 + \frac{hf}{k_B T} - 1} \frac{8\pi V}{c^3} f^2 \\
&= k_B T \frac{8\pi V}{c^3} f^2
\end{aligned}$$

gives the classical result.

(b)

The classical formula grows as f^2 and diverges at high f. Planck's formula has f^3 in the numerator and $e^{\frac{hf}{k_B T}} - 1$ in the denominator. The exponential in the denominator will grow faster, and the limit of Planck's formula as $f \to \infty$ is 0 (can show this using L'Hopital's rule).

Harris 3.23

If 590nm is the cutoff at which no electrons are ejected from the metal plate, the energy of the photons is equivalent to the work function of the metal. When light of one-third the wavelength strikes the plate:

$$\begin{aligned}
\mathrm{KE} &= \frac{hc}{\lambda/3} - \phi \\
\mathrm{KE} &= \frac{3hc}{\lambda} - \frac{hc}{\lambda} \\
(\gamma - 1)m_e c^2 &= \frac{2hc}{\lambda} \\
\gamma &= 1 + \frac{2h}{\lambda m_e c} \\
\gamma = 1 + \frac{2 \times 6.63 \times 10^{-34}\mathrm{Js}}{590\mathrm{nm} \times 9.11 \times 10^{-31}\mathrm{kg} \times 0.3\mathrm{Gm/s}} \\
\beta &= 4.06 \times 10^{-3} \\
v &= 1.22\mathrm{Mm/s}
\end{aligned}$$

We could also use the classical formula for kinetic energy. For $\beta = 0.1$, it gives $< 1\%$ error.

$$\begin{aligned}
\frac{1}{2}m_e v^2 &= \frac{2hc}{\lambda} \\
v &= \sqrt{\frac{4hc}{\lambda m_e}} \\
v = \sqrt{\frac{4 \times 6.63 \times 10^{-34}\mathrm{Js} \times 3 \times 10^8\mathrm{m/s}}{590\mathrm{nm} \times 9.11 \times 10^{-31}\mathrm{kg}}} \\
v &= 1.22 \times 10^6\mathrm{m/s}
\end{aligned}$$

Harris 3.29

The shortest wavelength results when all of the kinetic energy of an electron is used to produce a photon:

$$\begin{aligned}
\frac{1}{2}m_e v^2 &= \frac{hc}{\lambda} \\
v &= \sqrt{\frac{2hc}{\lambda m_e}} \\
v = \sqrt{\frac{2 \times 6.63 \times 10^{-34}\mathrm{Js} \times 3 \times 10^8\mathrm{m/s}}{0.062\mathrm{nm} \times 9.11 \times 10^{-31}\mathrm{kg}}} \\
\beta &= 0.280 \\
v &= 8.39 \times 10^7\mathrm{m/s}
\end{aligned}$$

Here, the classical approximation for an electron's kinetic energy is not as accurate.

$$(\gamma - 1)m_e c^2 = \frac{hc}{\lambda}$$
$$\gamma = 1 + \frac{h}{\lambda m_e c}$$
$$\gamma = 1 + \frac{6.63 \times 10^{-34}\text{Js}}{0.062\text{nm} \times 9.11 \times 10^{-31}\text{kg} \times 0.3\text{Gm/s}}$$
$$\beta = 0.272$$
$$v = 8.15 \times 10^{7}\text{m/s}$$

Harris 3.31

Refer to Figure (3-8).

(a)

Using Equation (3-8), the direction of the scattered photon is:

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$
$$\cos\theta = 1 - \frac{(\lambda' - \lambda)m_e c}{h}$$
$$\cos\theta = 1 - \frac{(0.061\text{nm} - 0.057\text{nm}) \times 9.11 \times 10^{-31}\text{kg} \times 3 \times 10^{8}\text{m/s}}{6.63 \times 10^{-34}\text{Js}}$$
$$\theta = 130^o$$

The photon scatters at an angle of 130^o to the initial direction.

(b)

Rearrange the energy conservation equation (3-6) to find the speed of the electron:

$$\frac{hc}{\lambda} + m_e c^2 = \frac{hc}{\lambda'} + \gamma m_e c^2$$
$$\gamma = 1 + \frac{h}{m_e c}\frac{(\lambda' - \lambda)}{\lambda\lambda'}$$
$$\gamma = 1.00279$$
$$\beta = 0.0746$$

Then solve for the direction using (3-5):

$$\sin\phi = \frac{h\sin\theta}{\gamma m_e \beta c \lambda'}$$
$$\phi = 24.0^0$$

Harris 3.34

Refer to Figure (3-8). Start with the conservation of momentum (3-4,3-5) and energy (3-6) equations. There are three equations and three unknowns: λ, λ' and θ. We only need to find λ, the wavelength of the source. To eliminate θ, first rearrange the momentum conservation equations:

$$p_\lambda - p_e\cos\phi = p_{\lambda'}\cos\theta$$
$$p_e\sin\phi = p_{\lambda'}\sin\theta$$

Then square both sides of each equation, add them together, and rearrange:

$$p_\lambda^2 - 2p_\lambda p_e\cos\phi + p_e^2 = p_{\lambda'}^2$$
$$p_\lambda^2 - p_{\lambda'}^2 = 2p_\lambda p_e\cos\phi - p_e^2$$

Rearrange the energy conservation equation:

$$\frac{hc}{\lambda} + m_e c^2 = \frac{hc}{\lambda'} + \gamma m_e c^2$$
$$p_\lambda + m_e c = p_{\lambda'} + \gamma m_e c$$
$$p_\lambda - (\gamma-1)m_e c = p_{\lambda'}$$
$$p_\lambda^2 - 2p_\lambda(\gamma-1)m_e c + (\gamma-1)^2 m_e^2 c^2 = p_{\lambda'}^2$$
$$p_\lambda^2 - p_{\lambda'}^2 = 2p_\lambda(\gamma-1)m_e c - (\gamma-1)^2 m_e^2 c^2$$

By equating the right sides of the previous two results, eliminate λ' and the need to solve a quadratic equation:

$$\begin{aligned}
2p_\lambda p_e \cos\phi - p_e^2 &= 2p_\lambda(\gamma-1)m_e c - (\gamma-1)^2 m_e^2 c^2 \\
p_\lambda &= \frac{(p_e^2 - (\gamma-1)^2 m_e^2 c^2)}{2(p_e\cos\phi - (\gamma-1)m_e c)} \\
p_\lambda &= \frac{(\gamma^2\beta^2 m_e^2 c^2 - (\gamma-1)^2 m_e^2 c^2)}{2(\gamma\beta m_e c\cos\phi - (\gamma-1)m_e c)} \\
p_\lambda &= \frac{m_e c(\gamma^2\beta^2 - (\gamma-1)^2)}{2(\gamma\beta\cos\phi - (\gamma-1))} \\
\lambda &= \frac{2h(\gamma\beta\cos\phi - (\gamma-1))}{m_e c(\gamma^2\beta^2 - (\gamma-1)^2)} \\
\lambda &= \frac{h(\gamma\beta\cos\phi - (\gamma-1))}{m_e c(\gamma-1)}
\end{aligned}$$

Substitute $\beta = 0.15$, m_e, h and $\cos\phi = 0.5$:

$$\lambda = 13.7\text{pm}$$

Harris 3.41

a

The momentum of the muon and antimuon is 0. A single photon cannot have 0 momentum.

b

To conserve momentum, the two photons must have opposite directions of motion and equal wavelengths. Find their wavelengths by equating the total energies of the muons and photons:

$$\begin{aligned}
2m_\mu c^2 &= 2\frac{hc}{\lambda} \\
\lambda &= \frac{h}{m_\mu c} \\
\lambda &= \frac{6.63\times10^{-34}\text{Js}}{1.88\times10^{-28}\text{kg}\times0.3\times10^{9}\text{m/s}} \\
\lambda &= 11.8\text{fm}
\end{aligned}$$

Harris 4.11

The probability of detecting an electron is proportional to the square of the absolute value of the wave function. Opening both slits is equivalent to doubling the wave function and the number of electrons detected would quadruple to 40.

Harris 4.21

a

For the neutron ($m = 1.67 \times 10^{-27}$kg) at 300K:

$$\lambda = \frac{h}{\sqrt{3mk_BT}} = 0.146\text{nm}$$

At $0.01c$:

$$\lambda = \frac{h}{p} = \frac{h}{mv} = 0.132\text{pm}$$

b

For the electron ($m = 9.11 \times 10^{-31}$kg) at 300K:

$$\lambda = \frac{h}{\sqrt{3mk_BT}} = 6.23\text{nm}$$

At $0.01c$:

$$\lambda = \frac{h}{p} = \frac{h}{mv} = 0.243\text{nm}$$

c

For this range of speeds, dimensions less than approximately 0.1nm - 0.1pm will reveal the wave nature of the neutron, and 6nm - 0.2nm for the electron.

Harris 4.27

a

The wavelength of the electrons is:

$$\frac{p^2}{2m} = qV$$
$$\lambda = \frac{h}{\sqrt{2mqV}}$$
$$\lambda = 0.275\text{nm}$$

$d = 0.010$mm is the distance between the slits and $L = 10$m the distance between the slits and the detectors. Define θ as the angle between the paths to detector X and the center detector and h as the distance between the two detectors. For destructive interference, the difference in the paths from the slits to detector X is one half of the wavelength. Using the small angle approximation:

$$\theta \approx \frac{h}{L} \approx \frac{\lambda/2}{d}$$
$$h = \frac{L\lambda}{2d}$$
$$h = 0.137\text{mm}$$

b

The amplitudes of the wave functions for each slit are proportional to the square root of the number of electrons detected. At the center detector there is constructive interference. Sum the amplitudes and square the result:

$$\Psi_1 \sim \sqrt{100} = 10$$
$$\Psi_2 \sim \sqrt{900} = 30$$
$$P_{1+2} \sim |\Psi_1 + \Psi_2|^2$$
$$P_{1+2} \sim |10 + 30|^2 = 1600$$

1600 electrons per second will be detected at the center detector.

c

At detector X there is destructive interference. Take the difference of the amplitudes and square the result:

$$P_X \sim |\Psi_1 - \Psi_2|^2$$
$$P_X \sim |10 - 30|^2 = 400$$

400 electrons per second will be detected at detector X.