Soutions to Extra Problems for PHY293 Modern Physics (December 2017)

4.73 $p = \frac{h}{\lambda} = \frac{h}{4R_{nuc}} = \frac{6.63\times10^{-34}\,\text{J·s}}{16\times10^{-15}\,\text{m}} = 4.14\times10^{-20}\text{kg·m/s}$. Classically the velocity would be $v = \frac{4.14\times10^{-20}\,\text{kg·m/s}}{9.11\times10^{-31}\,\text{kg}}$ $= 4.5\times10^{10}$ m/s. Impossible! An electron confined to such small dimensions will be moving relativistically.

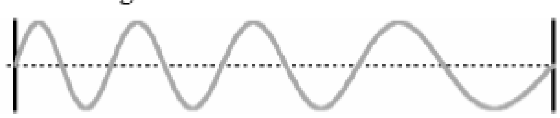
Relativistically, $p = 4.14\times10^{-20}\text{kg·m/s} = \gamma_u mu \rightarrow 4.14\times10^{-20}\text{kg·m/s} = (9.11\times10^{-31}\text{kg})\ \frac{u}{\sqrt{1-u^2/c^2}}$.

Squaring: $1.72\times10^{-39}\text{kg}^2\text{·m}^2/\text{s}^2 = 8.3\times10^{-61}\text{kg}^2\ \frac{u^2}{1-u^2/c^2}$ or $\frac{1.72\times10^{-39}\text{kg}^2\text{·m}^2/\text{s}^2}{8.3\times10^{-61}\text{kg}^2} = \frac{1}{1/u^2 - 1/c^2}$

$\Rightarrow u = 0.9999783c$. KE $= (\gamma_u - 1)mc^2 = \left(\frac{1}{\sqrt{1-(0.9999783)^2}} - 1\right)(9.11\times10^{-31}\text{kg})(9\times10^{16}\text{m}^2/\text{s}^2) = 1.2\times10^{-11}\text{J}$.

PE $= \frac{(9\times10^9\,\text{N·m}^2/\text{C}^2)(20\times1.6\times10^{-19}\,C)(-1.6\times10^{-19}\,C)}{4\times10^{-15}\,\text{m}} = -\ 1.2\times10^{-12}\text{J}$. The potential energy is by definition negative and is a maximum of zero when the electron is infinitely far from the nucleus. If at any point the electron has positive *total* (kinetic plus potential) energy, it may indeed make it out to infinitely far away. We have shown that the confined electron's KE would be *ten times* the potential energy—so the electron cannot be confined by this potential energy.

5.15 Its wavelength could not be constant. Since E is constant, the kinetic energy increases on the left, meaning a larger momentum and a smaller wavelength there.

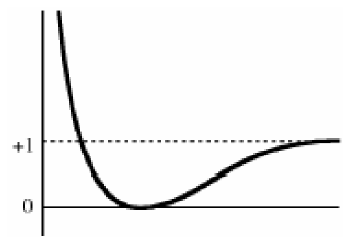


5.21 As $x \to 0$ this goes to $+\infty$; the $1/x^2$ diverges faster in the positive direction than $1/x$ in the negative. As $x \to \infty$ it goes to 1 $\frac{d}{dx}U(x) = -2\frac{1}{x^3} + \frac{2}{x^2}$. Setting this to 0 gives $x = \infty$ and $x = 1$. There must be a minimum at $x = 1$. $U(1) = 0$.

(b) When is KE zero? When $E = U$. $0.5 = \frac{1}{x^2} - \frac{2}{x} + 1 \to 0.5 - \frac{2}{x} + \frac{1}{x^2} = 0 \to 0.5x^2 - 2x + 1 \Rightarrow$

$x = \frac{2 \pm \sqrt{2^2 - 4(0.5)1}}{2(0.5)} = 2 \pm \sqrt{\mathbf{2}}$. With turning points on either side, **yes**, it would be bound.

(c) $2 = \frac{1}{x^2} - \frac{2}{x} + 1 \to -1 - \frac{2}{x} + \frac{1}{x^2} = 0 \to x^2 + 2x - 1 \Rightarrow x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)} = \frac{-2 \pm \sqrt{8}}{2}$. The only positive root is $\sqrt{\mathbf{2}} - \mathbf{1}$. **No.**



5.26 $E = \frac{\pi^2 \hbar^2 n^2}{2mL^2} = \frac{\pi^2 (1.055 \times 10^{-34}\,\mathrm{J{\cdot}s})^2 n^2}{2(1.67 \times 10^{-27}\,\mathrm{kg})(15 \times 10^{-15}\,\mathrm{m})^2} = 1.5 \times 10^{-13}\,\mathrm{J} \times n^2 \cong 1\mathrm{MeV} \times n^2$. Transitions between various n values should indeed generate photons whose energies are in the MeV range.

5.90 Set the probability integral over all space to unity. $1 = \frac{2}{\pi}a^3 \int_{-\infty}^{+\infty} \frac{1}{\left(x^2 + a^2\right)^2} dx$. The integral can be looked up in table, or done by trig substitution $x = a \tan\theta$. In any case, its value is $\frac{\pi}{2a^3}$, so the normalization constant is correct.

5.91 $\bar{x} = \frac{2}{\pi}a^3 \int_{-\infty}^{+\infty} x \frac{1}{\left(x^2+a^2\right)^2}dx$. This is the integral of an odd function of x over a symmetric interval, so it is zero, as we would expect by symmetry. $\overline{x^2} = \frac{2}{\pi}a^3 \int_{-\infty}^{+\infty} x^2 \frac{1}{\left(x^2+a^2\right)^2}dx$. The integral can be looked up in table, or done by trig substitution $x = a\tan\theta$. Its value is $\pi/2a$, so $\overline{x^2} = \frac{2a^3}{\pi}\frac{\pi}{2a} = a^2$. Thus, $\Delta x = \sqrt{\overline{x^2} - \bar{x}^2} = \boldsymbol{a}$.

5.92 It must solve the Schrödinger equation. $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + U(x)\psi(x) = 0$.

Thus $U(x) = \frac{1}{\psi(x)}\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)$. But $\frac{d^2}{dx^2}\psi(x) = \sqrt{\frac{2}{\pi}}a^{3/2}\frac{d}{dx}\frac{-2x}{(x^2+a^2)^2}$

$$= \sqrt{\frac{2}{\pi}}a^{3/2}\frac{-2(x^2+a^2)^2 - (-2x)(4x)(x^2+a^2)}{(x^2+a^2)^4} = \sqrt{\frac{2}{\pi}}a^{3/2}\frac{-2(x^2+a^2) - (-2x)(4x)}{(x^2+a^2)^3} = \sqrt{\frac{2}{\pi}}a^{3/2}\frac{6x^2-2a^2}{(x^2+a^2)^3}$$

Therefore, $U(x) = \frac{1}{\sqrt{2/\pi}a^{3/2}}(x^2+a^2)\frac{\hbar^2}{2m}\sqrt{\frac{2}{\pi}}a^{3/2}\frac{6x^2-2a^2}{(x^2+a^2)^3} = \frac{\hbar^2}{2m}\frac{6x^2-2a^2}{(x^2+a^2)^2}$

(c) We find the classical turning points by setting the potential energy equal to the total:

$U(x) = E \rightarrow \frac{\hbar^2}{2m}\frac{6x^2-2a^2}{(x^2+a^2)^2} = 0 \Rightarrow x = \pm a/\sqrt{3}$. Thus: $-\boldsymbol{a}/\sqrt{\mathbf{3}} < \mathbf{x} < +\boldsymbol{a}/\sqrt{\mathbf{3}}$